Pre-class Warm-up!!!

Find the best way to complete the sentence:

One Joule is

a. the force required to accelerate 1 gram at 1 cm/sec

b. the energy required to raise the temperature of 1 cubic centimeter of water by 1 degree centigrade

c. the work done in lifting 1 kilogram through 1 meter

d. the work done by 1 Newton moving a distance of 1 meter ✓

e. None of the above.

Maybe a Joule is a kind of candy?

Should Joule have a capital J?

Yes

No

7.1 The path integral 7.2 The line integral

We learn:

- Two closely related integrals along a path or curve
- How to set them up and evaluate them
- Physical interpretation
- Different notation for these integrals
- Orientation of a curve
- Theoretical things: independence of the parametrization of the curve (up to orientation for line integrals); Riemann sums.
- The line integral of a gradient vector field

We do not need to know:

- Total curvature in 7.1
- Simple curves, closed curves in 7.2
- You are not tested on the 'theoretical things'.

Types of question:

- mainly evaluate the integrals

The two kinds of integral

Both take a path c : [a,b] -> R^n
(In the book n = 3 .)
We want to assume $c'(t) \neq 0$ always.
Thus c traces along the curve from one end to the other, without retracing itself.

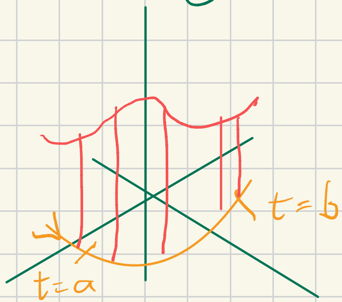
The orientation of c is the direction we travel along the curve.
There are two orientations.

For the path integral in 7.1 we take a scalar function f : R^n -> R and define

$$\int_c f\,ds = \int_a^b f(c(t))\|c'(t)\|\,dt.$$

If $f(x_1, \dots, x_n) = 1$ always this gives the arc length of c between $t=a$ and $t=b$.

More generally the integral gives the curvy area under the graph of f.



For the line integral in 7.2 we take a vector field F : R^n -> R^n and define

$$\int_c F \cdot d\underline{s} = \int_a^b F(c(t)) \cdot c'(t)\, dt$$

Notation, at the moment

Use for calculating.

$$= \int_c F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n$$

where $F = (F_1, \ldots, F_n)$

Think $dx_i = c_i'(t)\, dt$

where $c = (c_1, \ldots, c_n)$

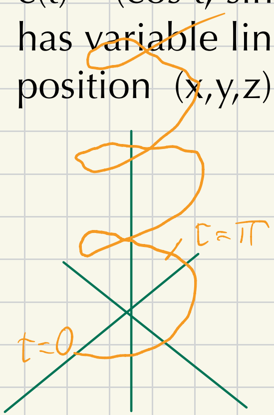
Physical interpretation : Work done by a vector field on a particle moving along the path c.

Questions in 7.1

These are either: find a parametrization of a given curve. Sometimes it is given in pieces and you have to parametrize it in pieces.
Or: evaluate the integral.

Bad question: sometimes this is negative

Like question 10.
A wire is bent into a helix parametrized by $c(t) = (\cos t, \sin t, t)$ where $0 \leq t \leq \pi$. The wire has variable line-density that is $xy + z$ at position (x,y,z). Find the mass of the wire.



The line density could be measured in g/cm

$c'(t) = (-\sin t, \cos t, 1)$

The mass is $\int_C (xy+z)\, ds$

$$= \int_0^\pi (xy+z)\|c'(t)\|\, dt$$

$$= \int_0^\pi (\cos t \sin t + t)\sqrt{(\sin t)^2 + (\cos t)^2 + 1^2}\, dt$$

$$= \left[\left(\frac{1}{2}\sin^2 t + \frac{t^2}{2}\right)\sqrt{2}\right]_0^\pi = \left(0 + \frac{\pi^2}{2} - 0 - 0\right)\sqrt{2}$$

$$= \frac{\pi^2\sqrt{2}}{2}$$

Physical interpretation:

Curvy area under graph, mass of wire.

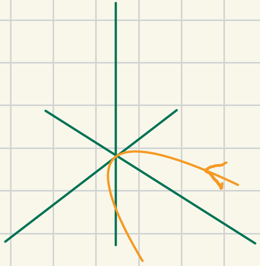
Questions in 7.2

These are: calculate the integral

Like Question 5:
Calculate the work done by the force field $F(x,y,z) = (y,-x,z)$ in moving a particle along the parabola $y = x^2$, $z = 0$ from $x = -1$ to $x = 2$.

Step 1 parametrize the path

$$c(t) = (t, t^2, 0), \quad c'(t) = (1, 2t, 0)$$



The work is

$$\int_c F \cdot d\underline{s} = \int_{-1}^{2} (y,-x,z) \cdot (1,2t,0)\, dt$$

$$= \int_{-1}^{2} (t^2, -t, 0) \cdot (1, 2t, 0)\, dt$$

$$= \int_{-1}^{2} (t^2 - 2t^2 + 0)\, dt = \left[-\frac{t^3}{3} \right]_{-1}^{2}$$

$$= -\frac{8}{3} - \frac{1}{3} = -3$$

Question: What does it mean if the work done by a force field is negative?

a. the question was wrong
b. the method of doing the question was wrong
c. energy was transferred from the particle ✓
d. energy was transferred to the particle
e. None of the above.

Theorems 1 and 2 of 7.2

Suppose c and p are two parametrizations of the same curve, with the same orientation. Then

$$\int_c F \cdot d\underline{s} = \int_p F \cdot d\underline{s}$$

If c and p have the opposite orientation then

$$\int_c F \cdot d\underline{s} = -\int_p F \cdot d\underline{s}$$

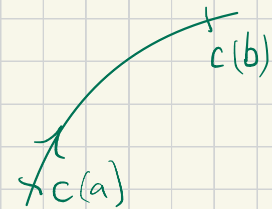
In either case

$$\int_c f\, ds = \int_p f\, ds$$

Theorem 3 of 7.2

Let f : R^n -> R and let c : [a,b] -> R^n be a path.

Then $\displaystyle\int_c \nabla f \cdot d\underline{s}$ =f(c(b)) - f(c(a)) .



Example: if $f : [a,b] \to \mathbb{R}$

then $\displaystyle\int \nabla f \cdot d\underline{s} = \int_a^b \frac{df}{dx}\, dx$

$= f(b) - f(a)$.

Proof. $\displaystyle\int_c \nabla f \cdot d\underline{s} = \int_a^b \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \cdot \left(c_1'(t), \dots, c_n'(t)\right) dt$

$\displaystyle = \int_a^b (Df)(c(t)) \cdot c'(t)\, dt = \int_a^b D\left[f(c(t))\right] dt$

$\displaystyle = \int_a^b \frac{d}{dt} f(c(t))\, dt = f(c(b)) - f(c(a))$.

Example

Find the work done by F(x,y) = (2xy, x^2) in moving along the path
c(t) = (√(1+t^2) , e^(sin t)) from t = 0 to t = 1.

Solution Observe that

$$F = (2xy, x^2) = \nabla f \text{ where}$$

$$f(x,y) = x^2 y$$

Work done is

$$\int_c F \cdot d\underline{s} = f(c(1)) - f(c(0))$$

$$= f(\sqrt{2}, e^{\sin(1)}) - f(1,1)$$

$$= 2e^{\sin(1)} - 1 \qquad \square$$